Local time and Tanaka formula for the G-Brownian Motion

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Abstract

In this paper, we study the notion of local time and Tanaka formula for the G-Brownian motion. Moreover, the joint continuity of the local time of the G-Brownian motion is obtained and its quadratic variation is proven. As an application, we generalize Itô's formula with respect to the G-Brownian motion to convex functions.

Keywords: G-expectation; G-Brownian motion; local time; Tanaka formula; quadratic variation.

1 Introduction

The objective of the present paper is to study the local time as well as the Tanaka formula for the G-Brownian motion in the framework of sublinear expectation spaces.

Motivated by uncertainty problems, risk measures and superhedging in finance, Peng has introduced a new notion of nonlinear expectation, the so-called G-expectation (see [14], [16], [17], [19], [20]), which is associated with the following nonlinear heat equation

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} = G\left(\dfrac{\partial^2 u(t,x)}{\partial x^2}\right), & (t,x) \in [0,\infty) \times \mathbb{R}, \\
u(0,x) = \varphi(x),
\end{cases}
$$

where, for given parameters $0 \le \underline{\sigma} \le \overline{\sigma}$, the sublinear function G is defined as follows:

$$
G(\alpha) = \frac{1}{2}(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-), \quad \alpha \in \mathbb{R}.
$$

The G-expectation represents a special case of general nonlinear expectations $\hat{\mathbb{E}}$ which importance stems from the fact that they are related to risk measures ρ in finance by the relation $\hat{\mathbb{E}}[X] = \rho(-X)$, where X runs the class of contingent claims. Although the G-expectations represent only a special case, their importance inside the class of nonlinear

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expectations stems from the stochastic analysis which it allows to develop, in particular, such important results as the law of large numbers and the central limit theorem under nonlinear expectations, obtained by Peng [15], [18] and [21].

Together with the notion of G-expectations Peng also introduced the related G-normal distribution and the G-Brownian motion. The G-Brownian motion is a stochastic process with stationary and independent increments and its quadratic variation process is, unlike the classical case of a linear expectation, a non deterministic process. Based on these both processes, an Itô calculus for the G-Brownian motion has been developed recently in [14], [16] and [17] and [10].

The fundamental notion of the local time for the classical Brownian motion has been introduced by Lévy in [9], and the first to prove its existence was Trotter [24] in 1958. In virtue of its various applications in stochastic analysis the notion of local time and the related Tanaka formula have been well studied by several authors since then, and it has also been extended to other classes of stochastic processes (see e.g., [3] and [22]).

It should be expected that the notion of local time and the very narrowly related Tanaka formula will have the same importance in the still very recent stochastic analysis on sublinear expectation spaces. However, unlike the study of the local time for the classical Brownian motion, its investigation with respect to the G-Brownian motion meets several difficulties: firstly, in contrast to the classical Brownian motion the G-Brownian motion is not defined on a given probability space but only on a sublinear expectation space. The G-expectation $\hat{\mathbb{E}}$ can be represented as the upper expectation of a subset of linear expectations $\{E_P, P \in \mathcal{P}\}$, *i.e.*,

$$\hat{\mathbb{E}}[\cdot] = \sup_{P \in \mathcal{P}} E_P[\cdot],$$

where P runs a large class of probability measures $\mathcal{P}$, which are mutually singular. Let us point out that this is a very common situation in financial models under volatility uncertainty (see [1], [5], [11]). Secondly, related with the novelty of the theory of G-expectations, there are a lot of tools in the classical Itô calculus which haven't yet been translated to the stochastic analysis on G-expectation spaces, among them there is, for instance, the dominated convergence theorem. In order to point out the difficulties, in section 3, ad hoc definition of the local time for the G-Brownian motion will be given independently of the probability measures $P \in \mathcal{P}$, and the limits of the classical stochastic analysis in the study of this local time will be indicated.

Our paper is organized as follows: Section 2 introduces the necessary notations and preliminaries and it gives a short recall of some elements of the G-stochastic analysis which will be used in what follows. In Sections 3, an intuitive approach to the local time shows its limits in the frame of the classical stochastic analysis and works out the problems to be studied. In Section 4, we define the notion of the local time for the G-Brownian motion and prove the related Tanaka formula, which non-trivially generalizes the classical one. Moreover, by using Kolmogorov's continuity criterion under the nonlinear expectation the existence of a jointly continuous version of the local time is shown. Finally, Section 5

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investigates the quadratic variation of the local time for the G-Brownian motion, while section 6 gives a generalization of the Itô formula for the G-Brownian motion to convex functions.

2   Notations and preliminaries

In this section, we introduce some notations and preliminaries of the theory of sublinear expectations and the related G-stochastic analysis, which will be needed in what follows. More details of this section can be found in Peng [14], [16], [17], [19] and [20].

Let Ω be a given nonempty set and $\mathcal{H}$ a linear space of real valued functions defined on Ω such that, $1 \in \mathcal{H}$ and $|X| \in \mathcal{H}$, for all $X \in \mathcal{H}$.

Definition 2.1 A Sublinear expectation $\hat{\mathbb{E}}$ *on* $\mathcal{H}$ *is a functional* $\hat{\mathbb{E}} : \mathcal{H} \mapsto \mathbb{R}$ *satisfying the following properties: for all* $X, Y \in \mathcal{H}$, *we have*

(i) **Monotonicity:** *If* $X \geq Y$, *then* $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.

(ii) **Preservation of constants:** $\hat{\mathbb{E}}[c] = c$, *for all* $c \in \mathbb{R}$.

(iii) **Subadditivity:** $\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]$.

(iv) **Positive homogeneity:** $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, *for all* $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called a sublinear expectation space.*

Remark 2.2 $\mathcal{H}$ *is considered as the space of random variables on* Ω.

Let us now consider a space of random variables $\mathcal{H}$ with the additional property of stability with respect to bounded and Lipschitz functions. More precisely, we suppose, if $X_i \in \mathcal{H}$, $i = 1, \cdots, d$, then

$$\varphi(X_1, \cdots, X_d) \in \mathcal{H}, \ \text{ for all } \varphi \in C_{b,Lip}(\mathbb{R}^d),$$

where $C_{b,Lip}(\mathbb{R}^d)$ denotes the space of all bounded and Lipschitz functions on $\mathbb{R}^d$.

Definition 2.3 *In a sublinear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, *a random vector* $Y = (Y_1, \cdots, Y_n), Y_i \in \mathcal{H}$, *is said to be independent under* $\hat{\mathbb{E}}$ *from another random vector* $X = (X_1, \cdots, X_m), X_i \in \mathcal{H}$, *if for each test function* $\varphi \in C_{b,lip}(\mathbb{R}^{m+n})$ *we have*

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Definition 2.4 *Let* X_1 *and* X_2 *be two* n–*dimensional random vectors defined respectively in the sublinear expectation spaces* $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ *and* $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. *They are called identically distributed, denoted by* $X_1 \sim X_2$, *if*

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)], \quad \text{for all } \varphi \in C_{b.Lip}(\mathbb{R}^n).$$

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After the above basic definition we introduce now the central notion of G-normal distribution.

Definition 2.5 *(G-normal distribution) Let be given two reals $\underline{\sigma}, \overline{\sigma}$ with $0 \le \underline{\sigma} \le \overline{\sigma}$. A random variable ξ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called $G_{\underline{\sigma},\overline{\sigma}}$-normal distributed,denoted by $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, if for each $\varphi \in C_{b,lip}(\mathbb{R})$, the following function defined by*

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}\xi)], \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

is the unique viscosity solution of the following parabolic partial differential equation :

$$\begin{cases} \partial_t u(t, x) = G(\partial_{xx}^2 u(t, x)), & (t, x) \in [0, \infty) \times \mathbb{R}, \\ u(0, x) = \varphi(x). \end{cases}$$

Here $G = G_{\underline{\sigma},\overline{\sigma}}$ is the following sublinear function parameterized by $\underline{\sigma}$ and $\overline{\sigma}$:

$$G(\alpha) = \frac{1}{2}(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-), \quad \alpha \in \mathbb{R}$$

(Recall that $\alpha^+ = max\{0, \alpha\}$ and $\alpha^- = -min\{0, \alpha\}$).

Definition 2.6 *A process $B = \{B_t, t \ge 0\} \subset \mathcal{H}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a G-Brownian motion if the following properties are satisfied:*

(i) $B_0 = 0$;

(ii) *For each $t, s \ge 0$, the difference $B_{t+s} - B_t$ is $\mathcal{N}(0, [\underline{\sigma}^2 s, \overline{\sigma}^2 s])$-distributed and is independent from $(B_{t_1}, \cdots, B_{t_n})$, for all $n \in \mathbb{N}$ and $0 \le t_1 \le \cdots \le t_n \le t$.*

Throughout this paper, we let $\Omega = C_0(\mathbb{R}^+)$ be the space of all real valued continuous functions $(\omega_t)_{t \in \mathbb{R}^+}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left[(\max_{t \in [0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1 \right], \ \omega_t^1, \omega_t^2 \in \Omega.$$

We denote by $\mathcal{B}(\Omega)$ the Borel σ-algebra on Ω. We also set, for each $t \in [0, \infty)$, $\Omega_t := \{\omega_{\cdot \wedge t} : \omega \in \Omega\}$ and $\mathcal{F}_t := \mathcal{B}(\Omega_t)$.

Moreover, we will work with the following spaces:

- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$ -measurable real valued functions on Ω;

- $L^0(\Omega_t)$: the space of all $\mathcal{B}(\Omega_t)$-measurable real valued functions on Ω;

- $L_b(\Omega)$: the space of all bounded elements in $L^0(\Omega)$;

- $L_b(\Omega_t)$: the space of all bounded elements in $L^0(\Omega_t)$.

In [14], a G-Brownian motion is constructed on a sublinear expectation space $(\Omega, \mathbb{L}_G^p(\Omega), \hat{\mathbb{E}})$, where $\mathbb{L}_G^p(\Omega)$ is the Banach space defined as closure of $\mathcal{H}$ with respect to the norm $\|X\|_p := \hat{\mathbb{E}}[|X|^p]^{1/p}, 1 \leq p < \infty$. In this space the coordinate process $B_t(\omega) = \omega_t$, $t \in [0, \infty)$, $\omega \in \Omega$, is a G-Brownian motion. Let us point that the space $C_b(\Omega)$ of the bounded continuous functions on Ω is a subset of $\mathbb{L}_G^1(\Omega)$. Moreover, there exists a weakly compact family $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[\cdot] = \sup_{P \in \mathcal{P}} E_P[\cdot]. \tag{1}$$

So we can introduce the Choquet capacity

$$\hat{c}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Definition 2.7 *A set $A \subset \Omega$ is called polar if $\hat{c}(A) = 0$. A property is said to hold "quasi-surely" (q.s.) if it holds outside a polar set.*

The family of probability measures $\mathcal{P}$ allows to characterize the space $\mathbb{L}_G^p(\Omega)$ as follows:

$$\mathbb{L}_G^p(\Omega) = \left\{ X \in L^0(\Omega) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty, \text{ and } X \text{ is } \hat{c}\text{-quasi surely continuous} \right\}.$$

We also introduce the following spaces, for all $p > 0$,

- $\mathcal{L}^p := \left\{ X \in L^0(\Omega) : \hat{\mathbb{E}}[|X|^p] = \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \right\}$;

- $\mathcal{N} := \left\{ X \in L^0(\Omega) : X = 0, \ \hat{c}\text{-quasi surely (q.s.).} \right\}$

Obviously, $\mathcal{L}^p$ and $\mathcal{N}$ are linear spaces and $\left\{ X \in L^0(\Omega) : \hat{\mathbb{E}}[|X|^p] = 0 \right\} = \mathcal{N}$, for all $p > 0$. We put $\mathbb{L}^p := \mathcal{L}^p/\mathcal{N}$. As usual, we won't make here the distinction between classes and their representatives.

The following three propositions can be consulted in [4] and [7] .

Proposition 2.8 *For every monotonically decreasing sequence $\{X_n\}_{n=1}^\infty$ of nonnegative functions in $C_b(\Omega)$, which converges to zero q.s. on Ω, it holds $\lim_{n \to \infty} \hat{\mathbb{E}}[X_n] = 0$.*

Proposition 2.9 *For all $p > 0$, we have*

1. *$\mathbb{L}^p$ is a Banach space with respect to the norm $\|X\|_p := \left(\hat{\mathbb{E}}[|X|^p] \right)^{\frac{1}{p}}$.*

2. *$\mathbb{L}_G^p$ is the completion of $C_b(\Omega)$ with respect to the norm $\|\cdot\|_p$.*

We denote by $\mathbb{L}_*^p(\Omega)$ the completion of $L_b(\Omega)$ with respect to the norm $\|\cdot\|_p$.

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Proposition 2.10 *For a given $p \in (0, +\infty]$, let $\{X_n\}_{n=1}^{\infty} \subset \mathbb{L}^p$ be a sequence converging to X in $\mathbb{L}^p$. Then there exists a subsequence (X_{n_k}) which converges to X quasi-surely in the sense that it converges to X outside a polar set.*

We now recall the definition of quadratic variation process of the G-Brownian motion. We use $\{0 = t_0 < t_1 \cdots < t_n = t\}$ to denote a partition of $[0, t]$ such that $\max\{t_{i+1} - t_i, 0 \leq i \leq n - 1\} \to 0$, as $n \to \infty$. Then quadratic variation process of the G-Brownian motion is defined as follows:

$$\langle B \rangle_t := \mathbb{L}_G^2 \lim_{n \to \infty} \sum_{i=0}^{n-1} [B_{t_{i+1}^n \wedge t} - B_{t_i^n \wedge t}]^2 = B_t^2 - 2 \int_0^t B_s dB_s.$$

$\langle B \rangle$ is continuous and increasing outside a polar set.

In [10], a generalized Itô integral and a generalized Itô formula with respect to the G-Brownian motion are discussed as follows:

For arbitrarily fixed $p \geq 1$ and $T \in \mathbb{R}_+$, we first consider the following set of step processes:

$$M_{b,0}(0, T) = \Big\{ \eta : \eta_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t), 0 = t_0 < \cdots < t_n = T,$$

$$\xi_j \in L_b(\Omega_{t_j}), j = 0, \cdots, n - 1, n \geq 1 \Big\}. \tag{2}$$

Definition 2.11 *For an $\eta \in M_{b,0}(0, T)$ of the form (2), the related Bochner integral is*

$$\int_0^T \eta_t(\omega) dt = \sum_{j=0}^{n-1} \xi_j(\omega)(t_{j+1} - t_j).$$

For each $\eta \in M_{b,0}(0, T)$, we set

$$\hat{\mathbb{E}}_T[\eta] := \frac{1}{T} \hat{\mathbb{E}}[\int_0^T \eta_t dt] = \frac{1}{T} \hat{\mathbb{E}}[\sum_{j=0}^{n-1} \xi_j(t_{j+1} - t_j)],$$

and we introduce the norm

$$\|\eta\|_{M_*^p(0,T)} = \Big(\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt] \Big)^{1/p}$$

on $M_{b,0}(0, T)$. With respect to this norm, $M_{b,0}(0, T)$ can be continuously extended to a Banach space.

Definition 2.12 *For each $p \geq 1$, we denote by $M_*^p(0, T)$ the completion of $M_{b,0}(0, T)$ under the norm*

$$\|\eta\|_{M_*^p(0,T)} = \Big(\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt] \Big)^{1/p}.$$

Definition 2.13 *For every $\eta \in M_{b,0}(0,T)$ of the form (2)*

$$\eta_t(\omega) = \sum_{j=0}^{n-1} \xi_j(\omega) 1_{[t_j,t_{j+1})}(t),$$

the Itô integral

$$I(\eta) = \int_0^T \eta_s dB_s := \sum_{j=0}^{n-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

Lemma 2.14 *The mapping $I : M_{b,0}(0,T) \to \mathbb{L}^2_*(\Omega_T)$ is a continuous, linear mapping. Thus it can be continuously extended to $I : M^2_*(0,T) \to \mathbb{L}^2_*(\Omega_T)$. Moreover, for all $\eta \in M^2_*(0,T)$, we have*

$$\hat{\mathbb{E}}[\int_0^T \eta_s dB_s] = 0, \tag{3}$$

$$\hat{\mathbb{E}}[(\int_0^T \eta_s dB_s)^2] \leq \overline{\sigma}^2 \hat{\mathbb{E}}[\int_0^T \eta_s^2 ds]. \tag{4}$$

Definition 2.15 *For each fixed $p > 0$, we denote by $\eta \in M^p_\omega(0,T)$ the space of stochastic processes $\eta = (\eta_t)_{t\in[0,T]}$ for which there exists a sequence of increasing stopping times $\{\sigma_m\}_{m=1}^\infty$ (which can depend on the process η), with $\sigma_m \uparrow T$, quasi-surely, such that $\eta 1_{[0,\sigma_m]} \in M^p_*(0,T)$ and*

$$\widehat{c}(\int_0^T |\eta_s|^p ds < \infty) = 1.$$

The generalized Itô formula is obtained in [10].

Theorem 2.16 *Let $\varphi \in C^2(\mathbb{R})$ and*

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B\rangle_s + \int_0^t \beta_s dB_s, \text{ for all } t \in [0,T],$$

where α, η in $M^1_\omega(0,T)$ and $\beta \in M^2_\omega(0,T)$. Then, for each $0 \leq t \leq T$, we have

$$\varphi(X_t) - \varphi(X_0) = \int_0^t \partial_x\varphi(X_u)\beta_u dB_u + \int_0^t \partial_x\varphi(X_u)\alpha_u du$$
$$+ \int_0^t [\partial_x\varphi(X_u)\eta_u + \frac{1}{2}\partial_{xx}^2\varphi(X_u)\beta_u^2]d\langle B\rangle_u.$$

Example 2.17 *For all $\varphi \in C^2(\mathbb{R})$ and $t \geq 0$, then we have*

$$\varphi(B_t) = \varphi(0) + \int_0^t \varphi_x(B_s)dB_s + \frac{1}{2}\int_0^t \varphi_{xx}(B_s)d\langle B\rangle_s.$$

3 An intuitive approach to the local time

We consider the G-Brownian motion B. Recall that B is continuous and a P-martingale, for all $P \in \mathcal{P}$ (see (1)), and its quadratic variation process $\langle B \rangle$ is continuous and increasing outside a polar set $\mathcal{N}$. Let us give a definition of the local time of B, which is independent of the underlying probability measure $P \in \mathcal{P}$. For all $a \in \mathbb{R}$, $t \in [0,T]$, and $\omega \in \Omega \setminus \mathcal{N}$,

$$L_t^a(\omega) = \begin{cases} \overline{\lim_{\varepsilon \downarrow 0}} \dfrac{1}{2\varepsilon} \int_0^t 1_{(a-\varepsilon,a+\varepsilon)}(B_s(\omega)) d\langle B \rangle_s(\omega), & \overline{\lim_{\varepsilon \downarrow 0}} \dfrac{1}{2\varepsilon} \int_0^t 1_{(a-\varepsilon,a+\varepsilon)}(B_s(\omega)) d\langle B \rangle_s(\omega) < \infty; \\ 0, & \text{otherwise.} \end{cases}$$

We now study L^a under each probability $P \in \mathcal{P}$. For this we consider the stochastic integral

$$M_t^P = \int_0^t sgn(B_s - a) dB_s, \ t \in [0,T], \tag{5}$$

under $P \in \mathcal{P}$. Indeed, under P the G-Brownian motion is a continuous square integrable martingale, so that the above stochastic integral under P is well-defined. We emphasize that the process M^P is defined only $P - a.s.$ Let $L^{a,P}$ be the local time associated with B under P, defined by the relation:

$$L_t^{a,P} = |B_t - a| - |a| - M_t^P, \ t \in [0,T].$$

It is well known that $L_t^{a,P}$ admits a P-modification that is continuous in (t,a), and

$$L_t^{a,P} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(a-\varepsilon,a+\varepsilon)}(B_s) d\langle B \rangle_s, P - a.s., \ t \in [0,T],$$

Thus, due to the definition of L^a, we have

$$L_t^a = L_t^{a,P}, P - a.s, t \in [0,T].$$

Consequently, L^a is has a continuous P-modification, for all $P \in \mathcal{P}$. However, P is not dominated by a probability measure. Thus the question, if we can find a continuous modification of L^a or even a jointly continuous modification of $(\alpha, t) \mapsto L_t^\alpha$ under $\hat{\mathbb{E}}$, i.e., w.r.t all $P \in \mathcal{P}$, is a nontrivial one which cannot be solved in the frame of the classical stochastic analysis. The other question which has its own importance is that of the integrability of $sgn(B. - a)$ in the framework of G-expectations. Indeed, in (5) we have considered the stochastic integral of $sgn(B. - a)$ under P, for each $P \in \mathcal{P}$ separately. However, the main difficulty is that the above family of probability measures is not dominated by one of these probability measures. To overcome this difficulty, we shall show that $sgn(B. - a)$ belongs to a suitable space of processes integrable w.r.t. to the G-Brownian motion B. It should be expected that this space is $M_*^2(0,T)$ (see Definition 2.12). This turns out to be correct if $\underline{\sigma} > 0$, but it is not clear at all in the case of $\underline{\sigma} = 0$. For this reason a larger

space $\widetilde{M}_*^2(0, T)$ (see Definition 4.1) is introduced in section 4, and its relationship with $M_*^2(0, T)$ is discussed. To get a jointly continuous modification of $(\alpha, t) \mapsto L_t^\alpha$ under $\hat{\mathbb{E}}$ in the framework of G-expectations, we use, in addition to the integrality of $sgn(B. - a)$ w.r.t. to the G-Brownian motion B, an approximation approach which is different from the classical method.

4 Local time and Tanaka formula for the G-Brownian motion

The objective of this section is to study the notion of local time and to obtain Tanaka formula for the G-Brownian motion. First, we shall generalize Itô integral with respect to the G-Brownian motion, which plays an important role in what follows.

For each $\eta \in M_{b,0}(0, T)$, we introduce the norm

$$||\eta||_{\widetilde{M}_*^p(0,T)} = \Big(\hat{\mathbb{E}}[\int_0^T |\eta_t|^p d\langle B \rangle_t] \Big)^{1/p}$$

on $M_{b,0}(0, T)$. With respect to this norm, $M_{b,0}(0, T)$ can be continuously extended to a Banach space.

Definition 4.1 *For each $p \geq 1$, we denote by $\widetilde{M}_*^p(0, T)$ the completion of $M_{b,0}(0, T)$ under the norm*

$$||\eta||_{\widetilde{M}_*^p(0,T)} = \Big(\hat{\mathbb{E}}[\int_0^T |\eta_t|^p d\langle B \rangle_t] \Big)^{1/p}.$$

Remark 4.2 *For every $p \geq 1$, it is easy to check that $M_*^p(0, T) \subset \widetilde{M}_*^p(0, T)$. Moreover, if $\underline{\sigma} > 0$, then $M_*^p(0, T) = \widetilde{M}_*^p(0, T)$. If $\underline{\sigma} = 0$, then we shall show in Remark 4.7 that $M_*^p(0, T)$ is a strict subset of $\widetilde{M}_*^p(0, T)$.*

For every $\eta \in M_{b,0}(0, T)$, the Itô integral with respect to the G-Brownian motion is defined in Definition 2.13.

Lemma 4.3 *The mapping $I : M_{b,0}(0, T) \to \mathbb{L}_*^2(\Omega_T)$ is a continuous, linear mapping. Thus it can be continuously extended to $I : \widetilde{M}_*^2(0, T) \to \mathbb{L}_*^2(\Omega_T)$. Moreover, for all $\eta \in \widetilde{M}_*^2(0, T)$, we have*

$$\hat{\mathbb{E}}[\int_0^T \eta_s dB_s] = 0,$$

$$\hat{\mathbb{E}}[(\int_0^T \eta_s dB_s)^2] \leq \hat{\mathbb{E}}[\int_0^T \eta_s^2 d\langle B \rangle_s].$$

Now we establish the Burkhölder-Davis-Gundy inequality for the framework of the G-stochastic analysis, which will be needed in what follows.

Lemma 4.4 *For each $p > 0$, there exists a constant $c_p > 0$ such that*
(i) for all $\eta \in M_*^2(0, T)$,

$$\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^{2p}] \leq c_p \hat{\mathbb{E}}[(\int_0^T \eta_s^2 d\langle B \rangle_s)^p] \leq \overline{\sigma}^{2p} c_p \hat{\mathbb{E}}[(\int_0^T \eta_s^2 ds)^p],$$

$$\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^{2p}] \geq \frac{1}{c_p} \hat{\mathbb{E}}[(\int_0^T \eta_s^2 d\langle B \rangle_s)^p] \geq \frac{\underline{\sigma}^{2p}}{c_p} \hat{\mathbb{E}}[(\int_0^T \eta_s^2 ds)^p];$$

(ii) for all $\eta \in \widetilde{M}_*^2(0, T)$,

$$\frac{1}{c_p} \hat{\mathbb{E}}[(\int_0^T \eta_s^2 d\langle B \rangle_s)^p] \leq \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^{2p}] \leq c_p \hat{\mathbb{E}}[(\int_0^T \eta_s^2 d\langle B \rangle_s)^p].$$

Proof. We only give the proof of (i), i.e., for $\eta \in M_*^2(0, T)$. The proof of (ii) is similar. Since for each $\alpha \in L_b(\Omega_t)$ we have

$$\hat{\mathbb{E}}[\alpha \int_t^T \eta_s dB_s] = 0,$$

then the process $\int_0^\cdot \eta_s dB_s$ is a P-martingale, for all $P \in \mathcal{P}$. Thus from the classical Burkhölder-Davis-Gundy inequality and the relation

$$\underline{\sigma}^2 t \leq \langle B \rangle_t \leq \overline{\sigma}^2 t, \ q.s,$$

we have, for all $P \in \mathcal{P}$,

$$E_P[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^{2p}] \leq c_p E_P[(\int_0^T \eta_s^2 d\langle B \rangle_s)^p] \leq \overline{\sigma}^{2p} c_p E_P[(\int_0^T \eta_s^2 ds)^p],$$

and

$$E_P[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^{2p}] \geq \frac{1}{c_p} E_P[(\int_0^T \eta_s^2 d\langle B \rangle_s)^p] \geq \frac{\underline{\sigma}^{2p}}{c_p} E_P[(\int_0^T \eta_s^2 ds)^p.]$$

We emphasize that the constant c_p coming from the classical Burkhölder-Davis-Gundy inequality, only depends on p but not on the underlaying probability measure P. Consequently, by taking the supremum over all $P \in \mathcal{P}$ we have

$$\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^{2p}] \leq c_p \hat{\mathbb{E}}[(\int_0^T \eta_s^2 d\langle B \rangle_s)^p] \leq \overline{\sigma}^{2p} c_p \hat{\mathbb{E}}[(\int_0^T \eta_s^2 ds)^p],$$

and

$$\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^{2p}] \geq \frac{1}{c_p} \hat{\mathbb{E}}[(\int_0^T \eta_s^2 d\langle B \rangle_s)^p] \geq \frac{\underline{\sigma}^{2p}}{c_p} \hat{\mathbb{E}}[(\int_0^T \eta_s^2 ds)^p].$$

The proof is complete. ∎

The following proposition is very important for our approach.

Proposition 4.5 *For any real a, all $\delta > 0$ and $t \geq 0$, we have*

$$\hat{\mathbb{E}}[\int_0^t 1_{[a,a+\delta]}(B_s)d\langle B\rangle_s] \leq C\delta.$$

Moreover, if $\underline{\sigma} > 0$, then we also have

$$\hat{\mathbb{E}}[\int_0^t 1_{[a,a+\delta]}(B_s)ds] \leq C\delta.$$

Here C is a constant which depends on t but not on δ neither on a.

Proof. For $\delta > 0$, we define the C^2- function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$, $|\varphi'(x)| \leq \delta^{-1}$, $x \in (\infty, a - \delta]$, and

$$\varphi''(x) = \begin{cases} 0; & if \ x \leq a - \delta, \\ \dfrac{x - a + \delta}{\delta^3}; & if \ a - \delta < x \leq a, \\ \dfrac{1}{\delta^2}; & if \ a < x \leq a + \delta, \\ -\dfrac{x - a - 2\delta}{\delta^3}; & if \ a + \delta < x \leq a + 2\delta, \\ 0; & if \ x \geq a + 2\delta. \end{cases}$$

Then we have $|\varphi'(x)| \leq 4\delta^{-1}$ and $\varphi''(x) \leq \delta^{-2}$.

By applying the generalized Itô formula for the G-Brownian motion (see Theorem 2.16) to $\varphi(B_t)$, we deduce that

$$\varphi(B_t) = \int_0^t \varphi'(B_s)dB_s + \frac{1}{2}\int_0^t \varphi''(B_s)d\langle B\rangle_s.$$

Therefore,

$$\begin{aligned}
\int_0^t 1_{[a,a+\delta]}(B_s)d\langle B\rangle_s &\leq \delta^2 \int_0^t \varphi''(B_s)d\langle B\rangle_s \\
&= 2\delta^2\varphi(B_t) - 2\delta^2 \int_0^t \varphi'(B_s)dB_s \\
&\leq 8\delta|B_t| - 2\delta^2 \int_0^t \varphi'(B_s)dB_s.
\end{aligned}$$

From the above inequalities and Lemma 2.14 it follows that

$$\begin{aligned}
\hat{\mathbb{E}}[\int_0^t 1_{[a,a+\delta]}(B_s)d\langle B\rangle_s] &\leq \hat{\mathbb{E}}[8\delta|B_t| - 2\delta^2 \int_0^t \varphi'(B_s)dB_s] \\
&\leq 8\delta\hat{\mathbb{E}}[|B_t|] + 2\delta^2\hat{\mathbb{E}}[-\int_0^t \varphi'(B_s)dB_s]
\end{aligned}$$

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$$= 8\delta\hat{\mathbb{E}}[|B_t|] = C\delta.$$

From Lemma 4.4 we have

$$\hat{\mathbb{E}}[\int_0^t 1_{[a,a+\delta]}(B_s)ds] \leq \frac{1}{\underline{\sigma}^2}\hat{\mathbb{E}}[\int_0^t 1_{[a,a+\delta]}(B_s)d\langle B\rangle_s] \leq C\delta.$$

The proof is complete. ∎

From the above proposition and Lemma 4.4, we can derive interesting results as follows:

Corollary 4.6 *For any real number a and $t \geq 0$, we have*

$$\int_0^t 1_{\{a\}}(B_s)d\langle B\rangle_s = 0, \quad q.s.$$

Moreover, if $\underline{\sigma} > 0$, then we have

$$\int_0^t 1_{\{a\}}(B_s)ds = 0, \quad q.s.$$

Remark 4.7 *If $\underline{\sigma} = 0$, then for every $p \geq 1$, $M_*^p(0,T)$ is a strict subset of $\widetilde{M}_*^p(0,T)$. In fact, by the above corollary we have, for any $\varepsilon \in (0,T)$,*

$$\hat{\mathbb{E}}[\int_0^T \frac{1}{s}1_{\{B_s=0\}}d\langle B\rangle_s] \leq \hat{\mathbb{E}}[\int_\varepsilon^T \frac{1}{s}1_{\{B_s=0\}}d\langle B\rangle_s] + \hat{\mathbb{E}}[\int_0^\varepsilon \frac{1}{s}1_{\{B_s=0\}}d\langle B\rangle_s]$$

$$\leq \frac{1}{\varepsilon}\hat{\mathbb{E}}[\int_0^T 1_{\{B_s=0\}}d\langle B\rangle_s] + \hat{\mathbb{E}}[\int_0^\varepsilon \frac{1}{s}1_{\{B_s=0\}}d\langle B\rangle_s]$$

$$= \hat{\mathbb{E}}[\int_0^\varepsilon \frac{1}{s}1_{\{B_s=0\}}d\langle B\rangle_s].$$

Letting $\varepsilon \downarrow 0$ we have

$$\hat{\mathbb{E}}[\int_0^T \frac{1}{s}1_{\{B_s=0\}}d\langle B\rangle_s] = 0,$$

i.e., $\{\frac{1}{s^p}1_{\{B_s=0\}}\}_{s\in[0,T]}$ can be approximately by 0 w.r.t. the norm in $\widetilde{M}_*^p(0,T)$, and so it belongs to $\widetilde{M}_*^p(0,T)$. On the other hand,

$$\hat{\mathbb{E}}[\int_0^T \frac{1}{s}1_{\{B_s=0\}}ds] = \sup_{u\in\mathcal{A}} E_P[\int_0^T \frac{1}{s}1_{\{\int_0^s u_s dW_s=0\}}ds]$$

$$= \int_0^T \frac{1}{s}ds = +\infty,$$

where W is a P-Brownian motion,

$\mathcal{A} := \{u : u \text{ is a } \mathcal{F}_t - \text{adapted process such that } 0 \leq u \leq \overline{\sigma}\}$,

$\mathcal{F}_t := \sigma\{B_s, 0 \leq s \leq t\} \vee \mathcal{N}$, $\mathcal{N}$ *is the collection of P-null sets,*

Therefore, $\{\frac{1}{s^p}1_{\{B_s=0\}}\}_{s\in[0,T]}$ can not belong to $M_*^p(0,T)$.

The following lemma will play an important role in what follows and its proof will be given after Theorem 4.9.

Lemma 4.8 *For each $a \in \mathbb{R}$, the process $sgn(B. - a) \in \widetilde{M}_*^2(0, T)$.*

Now we can state Tanaka formula for the G-Brownian motion as follows. For this we denote
$$sgn(x) = \begin{cases} 1; & x > 0, \\ 0; & x = 0, \\ -1; & x < 0. \end{cases}$$

Theorem 4.9 *For any real number a and all $t \geq 0$, we have*

$$|B_t - a| = |a| + \int_0^t sgn(B_s - a)dB_s + L_t^a,$$

where

$$L_t^a = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(a-\varepsilon, a+\varepsilon)}(B_s)d\langle B\rangle_s, \ (\lim \ in \ \mathbb{L}^2),$$

and L^a is an increasing process.
L^a is called the local time for G-Brownian motion at a.

Proof. Without loss of generality, we assume that $a = 0$. Let us define $\eta \in C^\infty(\mathbb{R})$ by putting
$$\eta(x) = \begin{cases} C \exp \left(\dfrac{1}{|x|^2 - 1} \right); & if \ |x| < 1, \\ 0; & if \ |x| \geq 1, \end{cases}$$
where C is the positive constant satisfying $\int_\mathbb{R} \eta(x)dx = 1$.
For every $n \in \mathbb{N}$, we put
$$\eta_n(x) := n\eta(nx), x \in \mathbb{R}.$$
Then $\eta_n \in C^\infty(\mathbb{R})$ and $\int_\mathbb{R} \eta_n(x)dx = 1$.
For any $\varepsilon > 0$, we set

$$\varphi_\varepsilon(x) = \begin{cases} \dfrac{1}{2}(\varepsilon + \dfrac{x^2}{\varepsilon}); & if \ |x| < \varepsilon, \\ |x|; & if \ |x| \geq \varepsilon. \end{cases}$$

Then, obviously

$$\varphi_\varepsilon'(x) = \begin{cases} \dfrac{x}{\varepsilon}; & if \ |x| \leq \varepsilon, \\ 1; & if \ x > \varepsilon, \\ -1; & if \ x < -\varepsilon, \end{cases}$$

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and

$$\varphi_\varepsilon''(x) = \begin{cases} \dfrac{1}{\varepsilon}; & if \ |x| < \varepsilon, \\ 0; & if \ |x| > \varepsilon \end{cases}$$

(φ'' is not defined at $-\varepsilon$ and ε).

Let us still introduce $\varphi_n := \varphi_\varepsilon * \eta_n$, i.e.,

$$\varphi_n(x) = \int_{\mathbb{R}} \eta_n(x - y)\varphi_\varepsilon(y)dy, x \in \mathbb{R}.$$

Then $\varphi_n(x) \in C^\infty(\mathbb{R})$, $0 \le \varphi_n'' \le \dfrac{1}{\varepsilon}$, $\varphi_n \to \varphi_\varepsilon, \varphi_n' \to \varphi_\varepsilon'$ uniformly in $\mathbb{R}$, and $\varphi_n'' \to \varphi_\varepsilon''$ pointwise (except at ε and $-\varepsilon$), as $n \to \infty$.

By applying the generalized Itô formula for the G-Brownian motion (see Theorem 2.16) to $\varphi_n(B_t)$, we deduce that

$$\varphi_n(B_t) = \varphi_n(0) + \int_0^t \varphi_n'(B_s)dB_s + \frac{1}{2}\int_0^t \varphi_n''(B_s)d\langle B\rangle_s. \qquad (6)$$

From $\varphi_n \to \varphi_\varepsilon, \varphi_n' \to \varphi_\varepsilon'$, uniformly in $\mathbb{R}$, it follows that

$$\varphi_n(B_t) \to \varphi_\varepsilon(B_t), \ \ \varphi_n(0) \to \varphi_\varepsilon(0),$$

and

$$\int_0^t \varphi_n'(B_s)dB_s \to \int_0^t \varphi_\varepsilon'(B_s)dB_s$$

in $\mathbb{L}^2$, as $n \to \infty$.

Setting

$$A_{n,\varepsilon} := (-\varepsilon - \frac{1}{n}, -\varepsilon + \frac{1}{n})\bigcup(\varepsilon - \frac{1}{n}, \varepsilon + \frac{1}{n}),$$

we observe that $\varphi_n'' = \varphi_\varepsilon''$ on $A_{n,\varepsilon}^c$.

From Corollary 4.6 we know that

$$\int_0^t 1_{\{-\varepsilon,\varepsilon\}}(B_s)d\langle B\rangle_s = 0, \ q.s.$$

Let us put $\varphi_\varepsilon''(x) = 0, x = \pm\varepsilon$.

Since $0 \le \varphi_n'' \le \dfrac{1}{\varepsilon}$, we have $|\varphi_n'' - \varphi_\varepsilon''| \le \dfrac{2}{\varepsilon}$ on $A_{n,\varepsilon}$. Therefore, we have

$$\hat{\mathbb{E}}[|\int_0^t \varphi_n''(B_s)d\langle B\rangle_s - \int_0^t \varphi_\varepsilon''(B_s)d\langle B\rangle_s|]$$

$$\le \ \hat{\mathbb{E}}[\int_0^t |\varphi_n''(B_s) - \varphi_\varepsilon''(B_s)|d\langle B\rangle_s]$$

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$$\leq \frac{2}{\varepsilon}\hat{\mathbb{E}}[\int_0^t 1_{A_{n,\varepsilon}}(B_s)d\langle B\rangle_s]$$

$$\leq \frac{2}{\varepsilon}\hat{\mathbb{E}}[\int_0^t 1_{(-\varepsilon-\frac{1}{n},-\varepsilon+\frac{1}{n})}(B_s)d\langle B\rangle_s] + \frac{2}{\varepsilon}\hat{\mathbb{E}}[\int_0^t 1_{(\varepsilon-\frac{1}{n},\varepsilon+\frac{1}{n})}(B_s)d\langle B\rangle_s].$$

From Proposition 4.5 we conclude that

$$\hat{\mathbb{E}}[|\int_0^t \varphi_n''(B_s)d\langle B\rangle_s - \int_0^t \varphi_\varepsilon''(B_s)d\langle B\rangle_s|] \leq \frac{C}{\varepsilon n} \to 0, \text{ as } n \to \infty.$$

Therefore, letting $n \to \infty$ in (6), we conclude that

$$\varphi_\varepsilon(B_t) = \frac{1}{2}\varepsilon + \int_0^t \varphi_\varepsilon'(B_s)dB_s + \frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(B_s)d\langle B\rangle_s. \tag{7}$$

From the definition of φ_ε it follows that

$$\hat{\mathbb{E}}[(\varphi_\varepsilon(B_t) - |B_t|)^2]$$
$$\leq \hat{\mathbb{E}}[(\varphi_\varepsilon(B_t) - |B_t|)^2 1_{|B_t|\geq\varepsilon}] + \hat{\mathbb{E}}[(\varphi_\varepsilon(B_t) - |B_t|)^2 1_{|B_t|<\varepsilon}]$$
$$= \hat{\mathbb{E}}[(\varphi_\varepsilon(B_t) - |B_t|)^2 1_{|B_t|<\varepsilon}] \leq \varepsilon^2 \to 0, \text{ as } \varepsilon \to 0.$$

By virtue of Lemma 4.4, we know that

$$\hat{\mathbb{E}}[|\int_0^t \varphi_\varepsilon'(B_s)dB_s - \int_0^t sgn(B_s)dB_s|^2]$$

$$\leq C\hat{\mathbb{E}}[\int_0^t (\varphi_\varepsilon'(B_s) - sgn(B_s))^2 d\langle B\rangle_s]$$

$$\leq C\hat{\mathbb{E}}[\int_0^t 1_{(-\varepsilon,\varepsilon)}(B_s)d\langle B\rangle_s],$$

and Proposition 4.5 allows to conclude that

$$\hat{\mathbb{E}}[|\int_0^t \varphi_\varepsilon'(B_s)dB_s - \int_0^t sgn(B_s)dB_s|^2] \to 0, \text{ as } \varepsilon \to 0.$$

Finally, from equation (7) it follows that

$$|B_t| = \int_0^t sgn(B_s)dB_s + L_t^0.$$

The proof is complete. ∎

The proof of Lemma 4.8: Now we prove that $sgn(B.) \in \widetilde{M}_*^2(0,T)$. Let $\pi = \{0 = t_0 < t_1 \cdots < t_n = T\}$, $n \geq 1$, be a partition of $[0,T]$ and $B_t^n = \sum_{j=0}^{n-1} B_{t_j} 1_{[t_j,t_{j+1})}(t)$. Then

$$\hat{\mathbb{E}}[\int_0^t (\varphi_n'(B_s) - \varphi_n'(B_s^n))^2 d\langle B\rangle_s]$$

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$$\leq \quad \frac{1}{\varepsilon^2}\hat{\mathbb{E}}[\int_0^t (B_s - B_s^n)^2 d\langle B\rangle_s] \to 0, \ as \ n \to \infty.$$

Since $\varphi_n' \to \varphi_\varepsilon'$, uniformly in $\mathbb{R}$, we have

$$\hat{\mathbb{E}}[\int_0^t (\varphi_\varepsilon'(B_s) - \varphi_n'(B_s))^2 d\langle B\rangle_s] \to 0, \ as \ n \to \infty.$$

By virtue of Proposition 4.5 we know that

$$\hat{\mathbb{E}}[\int_0^t (\varphi_\varepsilon'(B_s) - sgn(B_s))^2 d\langle B\rangle_s]$$

$$\leq \quad C\hat{\mathbb{E}}[\int_0^t 1_{(-\varepsilon,\varepsilon)}(B_s)d\langle B\rangle_s] \to 0, \ as \ \varepsilon \to 0.$$

Consequently, from the above estimates

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\int_0^t (sgn(B_s) - \varphi_n'(B_s^n))^2 d\langle B\rangle_s] \to 0, \ as \ \varepsilon \to 0.$$

Remark 4.10 *Similar to the proof of the above theorem, we can obtain* $1_{\{B_. > a\}} \in \widetilde{M}_*^2(0,T)$ *and* $1_{\{B_. \leq a\}} \in \widetilde{M}_*^2(0,T)$.

Remark 4.11 *In analogy to Theorem 4.9, we obtain two other forms of Tanaka formula:*

$$(B_t - a)^+ = (-a)^+ + \int_0^t 1_{\{B_s > a\}} dB_s + \frac{1}{2}L_t^a,$$

and

$$(B_t - a)^- = (-a)^- + \int_0^t 1_{\{B_s \leq a\}} dB_s + \frac{1}{2}L_t^a.$$

In [4], Denis, Hu and Peng obtained an extension of Kolmogorov continuity criterion to the framework of nonlinear expectation spaces. It will be needed for the study of joint continuity of local time for the G-Brownian motion L_t^a in (t, a).

Lemma 4.12 *Let* $d \geq 1$, $p > 0$ *and* $(X_t)_{t\in[0,T]^d} \subseteq \mathbb{L}^p$ *be such that there exists positive constants* C *and* $\varepsilon > 0$ *such that*

$$\hat{\mathbb{E}}[|X_t - X_s|^p] \leq C|t - s|^{d+\varepsilon}, \ for \ all \ t, s \in [0,T].$$

Then the field $(X_t)_{t\in[0,T]^d}$ *admits a continuous modification* $(\tilde{X})_{t\in[0,T]^d}$ *(i.e.* $\tilde{X}_t = X_t$, *q.s., for all* $t \in [0,T]$*) such that*

$$\hat{\mathbb{E}}\left[(\sup_{t\neq s} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\alpha})^p\right] < +\infty,$$

for every $\alpha \in [0, \varepsilon/p[$. *As a consequence, paths of* $\tilde{X}$ *are quasi-surely Hölder continuous of order* α, *for every* $\alpha < \varepsilon/p$.

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We will show that the local time for the G-Brownian motion has a jointly continuous modification. For the classical case, we refer to [2], [8] and [23]. For the proof we use an approximation method here, which is different from the classical case.

Theorem 4.13 *For all $t \in [0, T]$, then there exists a jointly continuous modification of $(a, t) \mapsto L_t^a$. Moreover, $(a, t) \mapsto L_t^a$ is Hölder continuous of order γ for all $\gamma < \frac{1}{2}$.*

Proof. From Lemma 4.4 we know that, for all $s, t \geq 0$ and $a, b \in \mathbb{R}$,

$$\hat{\mathbb{E}}[(|B_t - a| - |B_s - a|)^{2p}]$$
$$\leq C_p \hat{\mathbb{E}}[(|B_t - B_s|^{2p}] + C_p |a - b|^{2p}$$
$$\leq C_p |t - s|^p + |a - b|^{2p} \tag{8}$$

The constant C_p only depends on p. By choosing $p > 2$, we see from the generalized Kolmogorov continuity criterion (see Lemma 4.12) that $|B_t - a|$ has a jointly continuous modification in (a, t).

Let us now prove that also the integral $\int_0^t sgn(B_s - a)dB_s$ has a jointly continuous modification. For this end we let $\delta > 0$ and $p > 0$. Then

$$\hat{\mathbb{E}}[|\int_0^t sgn(B_s - a)dB_s - \int_0^t sgn(B_s - a - \delta)dB_s|^{2p}]$$
$$\leq 2^{2p} \hat{\mathbb{E}}[|\int_0^t 1_{[a,a+\delta]}(B_s)dB_s|^{2p}],$$

and from Lemma 4.4 it follows that

$$\hat{\mathbb{E}}[|\int_0^t sgn(B_s - a)dB_s - \int_0^t sgn(B_s - a - \delta)dB_s|^{2p}]$$
$$\leq C_p \hat{\mathbb{E}}[|\int_0^t 1_{[a,a+\delta]}(B_s)d\langle B \rangle_s|^p]. \tag{9}$$

We denote φ the same as in the proof of Proposition 4.5. By applying the generalized Itô formula for the G-Brownian motion (see Theorem 2.16) to $\varphi(B_t)$, we deduce that

$$\varphi(B_t) = \int_0^t \varphi'(B_s)dB_s + \frac{1}{2} \int_0^t \varphi''(B_s)d\langle B \rangle_s.$$

Therefore,

$$\int_0^t 1_{[a,a+\delta]}(B_s)d\langle B \rangle_s \leq \delta^2 \int_0^t \varphi''(B_s)d\langle B \rangle_s$$
$$= 2\delta^2 \varphi(B_t) - 2\delta^2 \int_0^t \varphi'(B_s)dB_s$$

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$$\leq \quad 8\delta|B_t| + 2\delta^2 |\int_0^t \varphi'(B_s)dB_s|. \tag{10}$$

Hereafter, C_p may be different from line to line, but only dependents on p. From the inequalities (9), (10) and Lemma 4.4 it follows that

$$\hat{\mathbb{E}}[|\int_0^t sgn(B_s - a)dB_s - \int_0^t sgn(B_s - a - \delta)dB_s|^{2p}]$$

$$\leq \quad C_p\hat{\mathbb{E}}[(8\delta|B_t| + 2\delta^2 |\int_0^t \varphi'(B_s)dB_s|)^p]$$

$$\leq \quad C_p\delta^p\hat{\mathbb{E}}[|B_t|^p] + C_p\delta^{2p}\hat{\mathbb{E}}[|\int_0^t \varphi'(B_s)dB_s|^p]$$

$$\leq \quad C_p\delta^p\hat{\mathbb{E}}[|B_t|^p] + C_p\delta^{2p}\hat{\mathbb{E}}[(\int_0^t (\varphi'(B_s))^2 d\langle B\rangle_s)^{\frac{p}{2}}]$$

$$\leq \quad C_p\delta^p\hat{\mathbb{E}}[|B_t|^p] + C_p\delta^p\hat{\mathbb{E}}[\langle B\rangle_t^{\frac{p}{2}}]$$

$$\leq \quad C_p\delta^p\hat{\mathbb{E}}[|B_T|^p] + C_p\delta^p\hat{\mathbb{E}}[\langle B\rangle_T^{\frac{p}{2}}]$$

$$= \quad C_p\delta^p. \tag{11}$$

Therefore, for all $0 \leq r \leq t$,

$$\hat{\mathbb{E}}[|\int_0^t sgn(B_s - a)dB_s - \int_0^r sgn(B_s - a - \delta)dB_s|^{2p}]$$

$$\leq \quad C_p\hat{\mathbb{E}}[|\int_0^r sgn(B_s - a)dB_s - \int_0^r sgn(B_s - a - \delta)dB_s|^{2p}]$$

$$\quad + C_p\hat{\mathbb{E}}[|\int_r^t sgn(B_s - a)dB_s|^{2p}]$$

$$\leq \quad C_p\delta^p + C_p\hat{\mathbb{E}}[(\int_r^t (sgn(B_s - a)^2 d\langle B\rangle_s)^p]$$

$$\leq \quad C_p\delta^p + C_p|t - r|^p. \tag{12}$$

We choose $p > 2$, then by the generalized Kolmogorov continuity criterion (Lemma 4.12), we obtain the existence of a jointly continuous modification of $(\alpha, t) \mapsto L_t^\alpha$. The proof is complete. ∎

Corollary 4.14 *For all $t \in [0, T], a \in \mathbb{R}$, we have*

$$\int_0^t sgn(B_s - a)dB_s, \int_0^t 1_{[a,\infty)}(B_s)dB_s, \int_0^t 1_{(-\infty,a)}(B_s)dB_s$$

have a jointly continuous modification.

5 Quadratic variation of local time for the G-Brownian motion

The objective of this section is to study the quadratic variation of the local time for the G-Brownian motion. For this end, we begin with the following Lemma:

Lemma 5.1 *If $f : \mathbb{R} \to [0, \infty)$ is a continuous function with compact support, then, for all $t \geq 0$, we have*

$$\int_{-\infty}^{\infty} f(a) \Big(\int_0^t sgn(B_s - a)dB_s \Big) da = \int_0^t \Big(\int_{-\infty}^{\infty} f(a)sgn(B_s - a)da \Big) dB_s, q.s.,$$

and

$$\int_{-\infty}^{\infty} f(a) \Big(\int_0^t 1_{[a,\infty)}(B_s)dB_s \Big) da = \int_0^t \Big(\int_{-\infty}^{\infty} f(a)1_{[a,\infty)}(B_s)da \Big) dB_s, q.s.$$

Proof. We only prove the fist inequality, the second inequality can be proved in the similar way. Without loss of generality we can assume that f has its support in $[0, 1]$. Let

$$\varphi_n(x) = \sum_{k=0}^{2^n-1} \frac{1}{2^n} f(\frac{k}{2^n}) sgn(x - \frac{k}{2^n}), x \in \mathbb{R}, n \geq 1.$$

Then

$$\int_0^t \varphi_n(B_s)dB_s = \sum_{k=0}^{2^n-1} \frac{1}{2^n} f(\frac{k}{2^n}) \int_0^t sgn(B_s - \frac{k}{2^n})dB_s.$$

By the proof Theorem 4.13 we know that the integral $\int_0^t sgn(B_s - a)dB_s$ is jointly continuous in (a, t). Therefore,

$$\int_0^1 \Big(\int_0^t f(a)sgn(B_s - a)dB_s \Big) da = \lim_{n \to \infty} \int_0^t \varphi_n(B_s)dB_s, \quad q.s. \tag{13}$$

From Lemma 4.4 and the convergence of $\varphi_n(x)$ to $\int_0^1 f(a)sgn(x - a)da$, uniformly w.r.t $x \in \mathbb{R}$, we obtain

$$\hat{\mathbb{E}}[| \int_0^t \varphi_n(B_s)dB_s - \int_0^t \Big(\int_0^1 f(a)sgn(B_s - a)da \Big) dB_s|^2]$$
$$\leq C\hat{\mathbb{E}}[\int_0^t \Big(\varphi_n(B_s) - \int_0^1 f(a)sgn(B_s - a)da \Big)^2 ds]$$
$$\to 0, \text{ as } n \to \infty.$$

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Using Proposition 2.10 we can deduce the existence of a subsequence $\{\varphi_{n_k}\}_{k=1}^{\infty}$ such that

$$\int_0^t \left(\int_0^1 f(a)sgn(B_s - a)da \right) dB_s = \lim_{k \to \infty} \int_0^t \varphi_{n_k}(B_s)dB_s, \quad q.s. \tag{14}$$

Finally, the relations (13) and (14) yield

$$\int_0^1 f(a) \left(\int_0^t sgn(B_s - a)dB_s \right) da = \int_0^t \left(\int_0^1 f(a)sgn(B_s - a)da \right) dB_s, \quad q.s.$$

The proof is complete. ∎

We now establish an occupation time formula. For the classical case, we refer to [2], [8] and [23].

Theorem 5.2 *For all $t \geq 0$ and all reals $a \leq b$, we have*

$$\int_0^t 1_{(a,b)}(B_s)d\langle B \rangle_s = \int_a^b L_t^x dx, \quad q.s.$$

Proof. Put $\varphi_{x,\varepsilon}(y) := \varphi_\varepsilon(y - x)$, where φ_ε is the function in the proof of Theorem 4.9. In analogy to inequality (7) we have

$$\varphi_{x,\varepsilon}(B_t) = \varphi_{x,\varepsilon}(B_0) + \int_0^t \varphi'_{x,\varepsilon}(B_s)dB_s + \frac{1}{2\varepsilon} \int_0^t 1_{(x-\varepsilon,x+\varepsilon)}(B_s)d\langle B \rangle_s, \tag{15}$$

where

$$\varphi'_{x,\varepsilon}(z) = \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} 1_{[y,\infty)}(z)dy - 1.$$

Thanks to Lemma 5.1, we have

$$
\begin{aligned}
\int_0^t \varphi'_{x,\varepsilon}(B_s)dB_s &= \frac{1}{\varepsilon} \int_0^t \int_{x-\varepsilon}^{x+\varepsilon} 1_{[y,\infty)}(B_s)dydB_s - B_t \\
&= \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \int_0^t 1_{[y,\infty)}(B_s)dB_sdy - B_t.
\end{aligned}
$$

Therefore, by equality (15) we have

$$\int_a^b \left(\varphi_{x,\varepsilon}(B_t) - \varphi_{x,\varepsilon}(B_0) - \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \int_0^t 1_{[y,\infty)}(B_s)dB_sdy + B_t \right)dx$$

$$= \frac{1}{2\varepsilon} \int_a^b \int_0^t 1_{(x-\varepsilon,x+\varepsilon)}(B_s)d\langle B \rangle_s dx. \tag{16}$$

According to Corollary 4.14, we obtain that

$$\lim_{\varepsilon \downarrow 0} \int_a^b \left(\varphi_{x,\varepsilon}(B_t) - \varphi_{x,\varepsilon}(B_0) - \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \int_0^t 1_{[y,\infty)}(B_s)dB_sdy + B_t \right)dx$$

$$= \int_a^b \left(|B_t - x| - |x| - 2 \int_0^t 1_{[x,\infty)}(B_s) dB_s + B_t \right) dx$$

$$= \int_a^b \left(|B_t - x| - |x| - \int_0^t sgn(B_s - x) dB_s \right) dx, \quad q.s. \tag{17}$$

For $z \in \mathbb{R}$, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_a^b 1_{(x-\varepsilon,x+\varepsilon)}(z) dx = 1_{(a,b)}(z) + \frac{1}{2} 1_{\{a\}}(z) + \frac{1}{2} 1_{\{b\}}(z).$$

Therefore,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_a^b \int_0^t 1_{(x-\varepsilon,x+\varepsilon)}(B_s) d\langle B \rangle_s dx$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \int_a^b 1_{(x-\varepsilon,x+\varepsilon)}(B_s) dx d\langle B \rangle_s$$

$$= \int_0^t 1_{(a,b)}(B_s) d\langle B \rangle_s + \frac{1}{2} \int_0^t 1_{\{a,b\}}(B_s) d\langle B \rangle_s, \quad q.s. \tag{18}$$

From Corollary 4.6 we know that

$$\int_0^t 1_{\{a,b\}}(B_s) d\langle B \rangle_s = 0, \quad q.s. \tag{19}$$

Consequently, from (26), (27), (28) and (29) it follows that

$$\int_a^b \left(|B_t - x| - |x| - \int_0^t sgn(B_s - x) dB_s \right) dx = \int_0^t 1_{(a,b)}(B_s) d\langle B \rangle_s, \quad q.s.$$

Finally, thanks to Theorem 4.9, we have

$$\int_a^b L_t^x dx = \int_0^t 1_{(a,b)}(B_s) d\langle B \rangle_s, \quad q.s.$$

The proof is complete. ∎

We now study the quadratic variation of the local time for the G-Brownian motion. For simplicity, we denote, for $t \geq 0$ and $x \in \mathbb{R}$,

$$Y_t^x = \int_0^t sgn(B_s - x) dB_s,$$
$$N_t^x = |B_t - x| - |x|.$$

Then, from Theorem 4.9 we have

$$L_t^x = N_t^x - Y_t^x.$$

We first give the following lemma, which is an immediate consequence of the proof of Theorem 4.13.

Lemma 5.3 *For every $p \geq 1$, there exists a constant $C = C(p) > 0$ such that*

$$\hat{\mathbb{E}}[|N_t^x - N_s^y|^p] \leq C(|x - y|^p + |t - s|^{\frac{p}{2}}),$$

$$\hat{\mathbb{E}}[|Y_t^x - Y_t^y|^p] \leq C|x - y|^{\frac{p}{2}},$$

for all $t, s \in [0, T]$ and $x, y \in \mathbb{R}$.

Now we give the main result in this section.

Theorem 5.4 *Let $\underline{\sigma} > 0$. Then for all $p \geq 1$ and $a \leq b$, we have along the sequence of partitions of $\pi_n = \{a_i^n = a + \dfrac{i(b-a)}{2^n}, \ i = 0, 1, \cdots, 2^n\}, n \geq 1$, of the interval of $[a, b]$ the following convergence*

$$\lim_{n \to \infty} \sum_{i=0}^{2^n - 1} (L_t^{a_{i+1}^n} - L_t^{a_i^n})^2 = 4 \int_a^b L_t^x dx, \ \text{ in } \mathbb{L}^p,$$

uniformly with respect to $t \in [0, T]$.

Proof. By applying the generalized Itô formula for the G-Brownian motion (cf. Theorem 2.16) we obtain

$$\sum_{i=0}^{2^n - 1} (L_t^{a_{i+1}^n} - L_t^{a_i^n})^2 - 4 \int_a^b L_t^x dx$$

$$= \sum_{i=0}^{2^n - 1} (N_t^{a_{i+1}^n} - N_t^{a_i^n} - Y_t^{a_{i+1}^n} + Y_t^{a_i^n})^2 - 4 \int_a^b L_t^x dx$$

$$= \sum_{i=0}^{2^n - 1} \Big((N_t^{a_{i+1}^n} - N_t^{a_i^n})^2 - 2(N_t^{a_{i+1}^n} - N_t^{a_i^n})(Y_t^{a_{i+1}^n} - Y_t^{a_i^n})$$

$$+ (Y_t^{a_{i+1}^n} - Y_t^{a_i^n})^2 \Big) - 4 \int_a^b L_t^x dx$$

$$= \sum_{i=0}^{2^n - 1} \Big((N_t^{a_{i+1}^n} - N_t^{a_i^n})^2 - 2(N_t^{a_{i+1}} - N_t^{a_i^n})(Y_t^{a_{i+1}^n} - Y_t^{a_i^n})$$

$$+ 4 \int_0^t (Y_s^{a_{i+1}^n} - Y_s^{a_i^n}) 1_{(a_i^n, a_{i+1}^n)}(B_s) dB_s + 4 \int_0^t 1_{(a_i^n, a_{i+1}^n)}(B_s) d\langle B \rangle_s \Big) - 4 \int_a^b L_t^x dx.$$

Thus, thanks to Theorem 5.2, we get

$$\sum_{i=0}^{2^n - 1} (L_t^{a_{i+1}^n} - L_t^{a_i^n})^2 - 4 \int_a^b L_t^x dx$$

$$= \sum_{i=0}^{2^n-1} (N_t^{a_{i+1}^n} - N_t^{a_i^n})^2 - 2 \sum_{i=0}^{2^n-1} (N_t^{a_{i+1}^n} - N_t^{a_i^n})(Y_t^{a_{i+1}^n} - Y_t^{a_i^n})$$

$$+4 \sum_{i=0}^{2^n-1} \int_0^t (Y_s^{a_{i+1}^n} - Y_s^{a_i^n}) 1_{(a_i^n, a_{i+1}^n)}(B_s) dB_s. \tag{20}$$

Then, from Lemma 5.3 and the subadditivity of the sublinear expectation $\hat{\mathbb{E}}[\cdot]$ it follows that

$$\hat{\mathbb{E}}[\Big(\sum_{i=0}^{2^n-1} (N_t^{a_{i+1}^n} - N_t^{a_i^n})^2 \Big)^p] \leq 2^{n(p-1)} \sum_{i=0}^{2^n-1} \hat{\mathbb{E}}[(N_t^{a_{i+1}^n} - N_t^{a_i^n})^{2p}]$$

$$\leq C 2^{n(p-1)} 2^n 2^{-2np} = C 2^{-np}. \tag{21}$$

Moreover, by virtue of Lemma 5.3, the subadditivity of the sublinear expectation $\hat{\mathbb{E}}[\cdot]$ and Hölder inequality for the sublinear expectation (see [17]) we can estimate the second term at the right hand of (20) as follows:

$$\hat{\mathbb{E}}[\Big(\sum_{i=0}^{2^n-1} (N_t^{a_{i+1}^n} - N_t^{a_i^n})(Y_t^{a_{i+1}^n} - Y_t^{a_i^n}) \Big)^p]$$

$$\leq 2^{n(p-1)} \sum_{i=0}^{2^n-1} \hat{\mathbb{E}}[|(N_t^{a_{i+1}^n} - N_t^{a_i^n})(Y_t^{a_{i+1}^n} - Y_t^{a_i^n})|^p]$$

$$\leq 2^{n(p-1)} \sum_{i=0}^{2^n-1} \Big(\hat{\mathbb{E}}[|N_t^{a_{i+1}^n} - N_t^{a_i^n}|^{2p}] \Big)^{\frac{1}{2}} \Big(\hat{\mathbb{E}}[|Y_t^{a_{i+1}^n} - Y_t^{a_i^n}|^{2p}] \Big)^{\frac{1}{2}}$$

$$\leq C 2^{n(p-1)} 2^n 2^{-np} 2^{-\frac{n}{2}p} = C 2^{-\frac{np}{2}}. \tag{22}$$

Here, for the latter estimate we have used Lemma 5.3.

Finally, let us estimate the third term at the right hand of (20). Using again the subadditivity of the sublinear expectation $\hat{\mathbb{E}}[\cdot]$ and Hölder inequality for the sublinear expectation as well as Lemma 4.4 we have

$$\hat{\mathbb{E}}[\Big(\sum_{i=0}^{2^n-1} \int_0^t (Y_t^{a_{i+1}^n} - Y_t^{a_i^n}) 1_{(a_i^n, a_{i+1}^n)}(B_s) dB_s \Big)^p]$$

$$\leq C \hat{\mathbb{E}}[\Big(\int_0^t \Big(\sum_{i=0}^{2^n-1} (Y_t^{a_{i+1}^n} - Y_t^{a_i^n}) 1_{(a_i^n, a_{i+1}^n)}(B_s) \Big)^2 ds \Big)^{\frac{p}{2}}]$$

$$\leq C \hat{\mathbb{E}}[\int_0^t \Big(\sum_{i=0}^{2^n-1} |Y_t^{a_{i+1}^n} - Y_t^{a_i^n}| 1_{(a_i^n, a_{i+1}^n)}(B_s) \Big)^p ds]$$

$$= C \hat{\mathbb{E}}[\int_0^t \sum_{i=0}^{2^n-1} |Y_t^{a_{i+1}^n} - Y_t^{a_i^n}|^p 1_{(a_i^n, a_{i+1}^n)}(B_s) ds]$$

$$\leq \quad C \sum_{i=0}^{2^n-1} \Big(\hat{\mathbb{E}}[\int_0^t |Y_t^{a_{i+1}^n} - Y_t^{a_i^n}|^{2p}ds]\Big)^{\frac{1}{2}} \Big(\hat{\mathbb{E}}[\int_0^t 1_{(a_i^n,a_{i+1}^n)}(B_s)ds]\Big)^{\frac{1}{2}}.$$

Consequently, thanks to Lemma 5.3 and Proposition 4.5, we have

$$\hat{\mathbb{E}}[\Big(\sum_{i=0}^{2^n-1} \int_0^t (Y_t^{a_{i+1}^n} - Y_t^{a_i^n})1_{(a_i^n,a_{i+1}^n)}(B_s)dB_s\Big)^p]$$

$$\leq C2^n 2^{-\frac{np}{2}} 2^{-\frac{n}{2}} = C2^{-\frac{n(p-1)}{2}}. \tag{23}$$

Finally, by substituting the estimates (21),(22) and (23) for the right hand of (20), we obtain

$$\hat{\mathbb{E}}[\Big(\sum_{i=0}^{2^n-1} (L_t^{a_{i+1}} - L_t^{a_i})^2 - 4\int_a^b L_t^x dx\Big)^p]$$

$$\leq C(2^{-np} + 2^{-\frac{np}{2}} + 2^{-\frac{n(p-1)}{2}}) \to 0, \text{ as } n \to \infty,$$

uniformly with respect to $t \in [0,T]$. The proof is complete. ∎

6   A generalized Itô formula for convex functions

In this section, our objective is to obtain a generalization of Itô's formula for the G-Brownian motion with help of the just proven Tanaka formula for the G-Brownian motion.

Given a convex function $f : \mathbb{R} \to \mathbb{R}$, we associate with the measure μ on $\mathcal{B}(\mathbb{R})$ which is defined as follows:

$$\mu[a,b) = f'_-(b) - f'_-(a), \text{ for all } a < b.$$

Then, for $\varphi \in C_K^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \varphi''(x)f(x)dx = \int_{\mathbb{R}} \varphi(x)\mu(dx).$$

This relation allows to identify the generalized second derivative of f with the measure μ.

Now we establish the main result in this section.

Theorem 6.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and assume that its right derivative f'_+ is bounded and $f'_-(B.) \in M_\omega^2(0,T)$. Then, for all $t \geq 0$,*

$$f(B_t) = f(0) + \int_0^t f'_-(B_s)dB_s + \int_{-\infty}^{\infty} L_t^a f''(da),$$

where $f'' = \mu$ is the measure introduced above.

Before giving the proof of Theorem 6.1, we establish the following lemmas, which will be needed later.

Lemma 6.2 If $f : \mathbb{R} \to [0, \infty)$ is a convex function and $\varphi \in C_b^1(\mathbb{R})$, then for all $t \geq 0$, we have

$$\int_{-\infty}^{\infty} \left(\int_0^t \varphi(B_s - a)dB_s \right) f''(da) = \int_0^t \left(\int_{-\infty}^{\infty} \varphi(B_s - a)f''(da) \right) dB_s.$$

Proof. We put

$$\Phi(x) = \int_0^x \varphi(y)dy, x \in \mathbb{R}.$$

Then, by applying the generalized Itô formula for the G-Brownian motion (cf. Theorem 2.16) to $\Phi(B_t - a)$, we deduce that

$$\Phi(B_t - a) = \Phi(-a) + \int_0^t \varphi(B_s - a)dB_s + \frac{1}{2} \int_0^t \varphi'(B_s - a)d \langle B \rangle_s.$$

Therefore,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left(\int_0^t \varphi(B_s - a)dB_s \right) f''(da) \\
&= \int_{-\infty}^{\infty} \Phi(B_t - a)f''(da) - \Phi(-a) \int_{-\infty}^{\infty} f''(da) - \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_0^t \varphi'(B_s - a)d \langle B \rangle_s \right) f''(da) \\
&= \int_{-\infty}^{\infty} \Phi(B_t - a)f''(da) - \Phi(-a) \int_{-\infty}^{\infty} f''(da) \\
&\quad - \frac{1}{2} \int_0^t \left(\int_{-\infty}^{\infty} \varphi'(B_s - a)f''(da) \right) d \langle B \rangle_s.
\end{aligned}
\tag{24}
$$

Let

$$g(x) = \int_{-\infty}^{\infty} \Phi(x - a)f''(da).$$

Then, using again the generalized Itô formula for the G-Brownian motion (see Theorem 2.16) to $g(B_t)$, we deduce that

$$g(B_t) = g(0) + \int_0^t g'(B_s)dB_s + \frac{1}{2} \int_0^t g''(B_s)d \langle B \rangle_s.$$

Consequently,

$$\int_0^t \left(\int_{-\infty}^{\infty} \varphi(B_s - a)f''(da) \right) dB_s = \int_{-\infty}^{\infty} \Phi(B_t - a)f''(da) - \Phi(-a) \int_{-\infty}^{\infty} f''(da)$$

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$$- \frac{1}{2} \int_0^t \left(\int_{-\infty}^{\infty} \varphi'(B_s - a) f''(da) \right) d \langle B \rangle_s. \qquad (25)$$

Finally, (24) and (25) allow to conclude that

$$\int_{-\infty}^{\infty} \left(\int_0^t \varphi(B_s - a) dB_s \right) f''(da) = \int_0^t \left(\int_{-\infty}^{\infty} \varphi(B_s - a) f''(da) \right) dB_s.$$

The proof is complete. ∎

Lemma 6.3 *If $f : \mathbb{R} \to [0, \infty)$ is a convex function and its right derivative f'_+ is bounded, then for all $t \geq 0$, we have*

$$\int_{-\infty}^{\infty} \left(\int_0^t sgn(B_s - a) dB_s \right) f''(da) = \int_0^t \left(\int_{-\infty}^{\infty} sgn(B_s - a) f''(da) \right) dB_s, q.s.$$

Proof. For $\varepsilon > 0$, we define C^1- function $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $|\varphi_\varepsilon| \leq 1$, and

$$\varphi_\varepsilon(x) = \begin{cases} -1; & if \ x \leq -\varepsilon, \\ 1; & if \ x \geq \varepsilon. \end{cases}$$

Then, we have $|\varphi_\varepsilon(x) - sgn(x)| \leq 2 1_{(-\varepsilon,\varepsilon)}(x)$, for $x \in \mathbb{R}$. From Lemma 6.2 we have

$$\int_{-\infty}^{\infty} \left(\int_0^t \varphi_\varepsilon(B_s - a) dB_s \right) f''(da) = \int_0^t \left(\int_{-\infty}^{\infty} \varphi_\varepsilon(B_s - a) f''(da) \right) dB_s.$$

From Lemma 4.4 and Proposition 4.5 it then follows that

$$\hat{\mathbb{E}}[| \int_0^t \left(\int_{-\infty}^{\infty} \varphi_\varepsilon(B_s - a) f''(da) \right) dB_s - \int_0^t \left(\int_{-\infty}^{\infty} sgn(B_s - a) f''(da) \right) dB_s |]$$

$$\leq \hat{\mathbb{E}}[\int_0^t \left(\int_{-\infty}^{\infty} \varphi_\varepsilon(B_s - a) f''(da) - \int_{-\infty}^{\infty} sgn(B_s - a) f''(da) \right)^2 d \langle B \rangle_s]^{\frac{1}{2}}$$

$$\leq \hat{\mathbb{E}}[\int_0^t \left(\int_{-\infty}^{\infty} |\varphi_\varepsilon(B_s - a) - sgn(B_s - a)| f''(da) \right)^2 d \langle B \rangle_s]^{\frac{1}{2}}$$

$$\leq C\hat{\mathbb{E}}[\int_0^t \left(\int_{-\infty}^{\infty} 1_{\{-\varepsilon,\varepsilon\}}(B_s - a) f''(da) \right)^2 d \langle B \rangle_s]^{\frac{1}{2}}$$

$$\leq C \left(\int_{-\infty}^{\infty} f''(da) \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \hat{\mathbb{E}}[\int_0^t 1_{\{-\varepsilon,\varepsilon\}}(B_s - a) d \langle B \rangle_s] f''(da) \right)^{\frac{1}{2}}$$

$$\leq C\varepsilon^{\frac{1}{2}} \int_{-\infty}^{\infty} f''(da) \to 0, \text{ as } \varepsilon \to 0,$$

and

$$\hat{\mathbb{E}}[| \int_{-\infty}^{\infty} \left(\int_0^t \varphi_\varepsilon(B_s - a) dB_s \right) f''(da) - \int_{-\infty}^{\infty} \left(\int_0^t sgn(B_s - a) dB_s \right) f''(da) |]$$

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$$\leq \int_{-\infty}^{\infty} \hat{\mathbb{E}}[|\int_0^t \Big(\varphi_\varepsilon(B_s - a) - sgn(B_s - a)\Big)dB_s|]f''(da)$$

$$\leq \int_{-\infty}^{\infty} \hat{\mathbb{E}}[|\int_0^t \Big(\varphi_\varepsilon(B_s - a) - sgn(B_s - a)\Big)^2 d\langle B\rangle_s||^{\frac{1}{2}} f''(da)$$

$$\leq C \int_{-\infty}^{\infty} \hat{\mathbb{E}}[\int_0^t 1_{\{-\varepsilon,\varepsilon\}}(B_s - a)d\langle B\rangle_s]^{\frac{1}{2}} f''(da)$$

$$\leq C\varepsilon^{\frac{1}{2}} \int_{-\infty}^{\infty} f''(da) \to 0, \text{ as } \varepsilon \to 0,$$

Consequently,

$$\int_{-\infty}^{\infty} \Big(\int_0^t sgn(B_s - a)dB_s\Big)f''(da) = \int_0^t \Big(\int_{-\infty}^{\infty} sgn(B_s - a)f''(da)\Big)dB_s, q.s.$$

The proof is complete. ∎

Now we give the proof of Theorem 6.1.

Proof. Since the function f'_+ is bounded, the measure f'' has its support in a compact interval I. From Section 3 in Appendix in [23], we know that

$$f(x) = \frac{1}{2}\int_I |x - a|f''(a)da + \alpha_I x + \beta_I, \tag{26}$$

where α_I and β_I are constants. Moreover, for all x outside of a countable set Γ in $\mathbb{R}$,

$$f'_-(x) = \frac{1}{2}\int_I sgn(x - a)|f''(da) + \alpha_I. \tag{27}$$

However, due to Corollary 4.6,

$$\int_0^t 1_\Gamma(B_s)d\langle B\rangle_s = 0, \ q.s., t \geq 0.$$

Consequently, thanks to the equalities (26) and (27), for all $t \geq 0$, we have

$$f(B_t) = \frac{1}{2}\int_I |B_t - a|f''(da) + \alpha_I B_t + \beta_I, \ q.s., \tag{28}$$

and

$$f'_-(B_t) = \frac{1}{2}\int_I sgn(B_t - a)|f''(da) + \alpha_I, \ q.s. \tag{29}$$

On the other hand, from Theorem 4.9 and equality (28) we deduce that

$$f(B_t) = \frac{1}{2}\int_I \Big(|a| + \int_0^t sgn(B_s - a)dB_s + L_t^a\Big)f''(da) + \alpha_I B_t + \beta_I$$

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$$= f(0) + \frac{1}{2} \int_I \left(\int_0^t sgn(B_s - a)dB_s \right) f''(da) + \frac{1}{2} \int_I L_t^a f''(da) + \alpha_I B_t.$$

Finally, thanks to Lemma 6.3 and equality (29), we have

$$\begin{aligned} f(B_t) &= f(0) + \frac{1}{2} \int_0^t \left(\int_I sgn(B_s - a)f''(da) \right) dB_s + \frac{1}{2} \int_I L_t^a f''(da) + \alpha_I B_t \\ &= f(0) + \int_0^t f'_-(B_s)dB_s + \int_I L_t^a f''(da). \end{aligned}$$

The proof is complete. ∎

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